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Embeddings of maximum packings in triple systems☆

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Dedicated to Curt Lindner on the occasion of his 65th birthday.

Abstract

Let $\mathrm{MPT}(v,\lambda)$ denote a maximum packing of triples of order v with index λ and $\mathrm{TS}(u,\lambda)$ denote a triple system of order u with index λ. For $\lambda > 1$ and $v \geqslant 6$, it is proved in this paper that the necessary and sufficient conditions for the embedding of an $\mathrm{MPT}(v,\lambda)$ in a $\mathrm{TS}(u,\lambda)$ are $\lambda(u-1) \equiv 0 \pmod 2$, $\lambda u(u-1) \equiv 0 \pmod 6$ and $u \geqslant 2v+1$.

Keywords: Maximum packing; Difference triple; Embedding

1. Introduction

Let V be a v-set, B a collection of 3-subsets (called blocks or triples) of V. A pair (V,B) is called a partial triple system of order v with index λ and denoted by $\mathrm{PTS}(v,\lambda)$ if each 2-subset of V is contained in at most λ triples.

Let (V,B) be a partial triple system of order v and index λ. (V,B) is called a maximum partial triple system if $|B| \geqslant |C|$ for any partial triple system $\mathrm{PTS}(v,\lambda)$ (V,C). A maximum partial triple system of order v and index λ is also called a maximum packing of triples (or simply a maximum packing) of order v with index λ and denoted by $\mathrm{MPT}(v,\lambda)$.

Let (V,B) be an $\mathrm{MPT}(v,\lambda)$, the leave of (V,B), denoted by $L(v,\lambda)$, is a multigraph (V,E) where an edge $\{x,y\} \in E$ with multiplicity m if and only if the corresponding 2-subset $\{x,y\}$ is contained in exactly $\lambda - m$ triples of B. It is well-known (see, e.g. [2]) that the leave of an $\mathrm{MPT}(v,\lambda)$ is empty if and only if $\lambda(v-1) \equiv 0 \pmod 2$ and $\lambda v(v-1) \equiv 0 \pmod 6$. In this case, the $\mathrm{MPT}(v,\lambda)$ is called a triple system and denoted by $\mathrm{TS}(v,\lambda)$. For fixed λ, if there exists a $\mathrm{TS}(v,\lambda)$, then v is called λ-admissible.

For $v \geqslant 6$, by Mendelsohn et al. [11], the only graphs which can be leaves of an $\mathrm{MPT}(v,\lambda)$ are shown in Table 1 (where v and λ are reduced modulo 6) with the following abbreviations:

Graphs of odd degrees

1F	a matching on v vertices
1FY	a matching on $v-4$ vertices and a tree on 4 vertices with one vertex of degree 3

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Table 1
Leaves of maximum packings

$\lambda \setminus v$	0	1	2	3	4	5
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	$1F$	$\emptyset$	$1F$	$\emptyset$	$1FY$	E_4
2	$\emptyset$	$\emptyset$	2	$\emptyset$	$\emptyset$	2
3	$1F$	$\emptyset$	06	$\emptyset$	$1FY$	$\emptyset$
4	$\emptyset$	$\emptyset$	E_4	$\emptyset$	$\emptyset$	E_4
5	$1F$	$\emptyset$	$1FY$	$\emptyset$	$1FY$	2

06(a)	$1FH$	a matching on $v-6$ vertices and a graph induced by AB, BC, BD, DF, DG
(b)	$1F_5$	a matching on $v-6$ vertices and a tree on 6 vertices with one vertex of degree 5
(c)	$1FYY$	a matching on $v-8$ vertices and two vertex-disjoint trees each on 4 vertices with one vertex of degree 3
(d)	$1F_3$	a matching on $v-2$ vertices and a triple edge AB, AB, AB
(e)	$1F_{-0-}$	a matching on $v-4$ vertices and a graph induced by AB, BC, BC, CD

Graphs of even degrees

2			a double edge AB, AB
E_4	(a)	C_4	a 4-cycle
	(b)	4	a quadruple edge AB, AB, AB, AB
	(c)	2^2	2 double edges AB, AB, CD, CD
	(d)	∞	AB, AB, BC, BC

Now let (X,A) be a partial triple system. (X,A) is said to be embedded in a partial triple system (Y,B) provided that $X \subset Y$ and $A \subseteq B$. We also say that (X,A) is a subsystem of (Y,B). The embedding problem is one of the fundamental problems in design theory. For the embeddings of triple systems, we have the following result:

Theorem 1.1 (Doyen and Wilson [5], Stern [15]). *Let u, v be λ-admissible and $v \geqslant 3$. Then a TS(v,λ) can be embedded in a TS(u,λ) if and only if $u \geqslant 2v+1$.*

Over the last 30 years much effort has been focussed on proving a similar theorem for embedding any PTS(v,λ) in a TS(u,λ). The best results to date are the following two theorems:

Theorem 1.2 (Rodger and Stubbs [13]). *Let u be odd λ-admissible. Then a PTS(v,λ) can be embedded in a TS(u,λ) if $u \geqslant 4v+1$.*

Theorem 1.3 (Hilton and Rodger [9]). *Let u be λ-admissible and $4 \mid \lambda$. Then a PTS(v,λ) can be embedded in a TS(u,λ) if $u \geqslant 2v+1$.*

The embedding problem for maximum packings has been studied extensively [6,7,10]. For $\lambda=1$, this embedding problem has been completely solved.

Theorem 1.4 (Mendelsohn and Rosa [10]; Hartman et al. [8]; Hartman [7]; Fu et al. [6]). *Let $u > v$. Any MPT$(v,1)$ can be embedded in an MPT$(u,1)$ if and only if*

(1) *if $v=6$ then $u=7$ or $u \geqslant 10$,*
(2) *if $v>6$ and v is even then $u=v+1$ or $u \geqslant 2v$, and*
(3) *if $v>6$ and v is odd then $u \geqslant 2v$.*

An MPT(v,λ) is called simple if it contains no repeated triples. In 1995, Milici et al. [12] proved that for $v \geqslant 3$ and any even λ, a simple MPT(v,λ) can be embedded in a simple MPT(u,λ) if and only if $u \geqslant 2v+1$. As a consequence of this result, we then have the following theorem.

Theorem 1.5 (Milici et al. [12]). *Let $v \geqslant 6$ and $\lambda \equiv 0 \pmod 2$. An MPT(v,λ) can be embedded in an MPT(u,λ) if and only if $u \geqslant 2v+1$.*

For $\lambda > 1$, it is easy to prove the following lemma by Table 1 and simple counting.

Lemma 1.6. *Let u be λ-admissible, $v \geqslant 6$ and $\lambda > 1$. Then a necessary condition for the embedding of an MPT(v,λ) in a TS(u,λ) is $u \geqslant 2v+1$.*

In this paper, by proving the following theorem, we show that the necessary condition of Lemma 1.6 is also sufficient for the embedding of any MPT(v,λ) in a TS(u,λ) for $\lambda > 1$.

Theorem 1.7. *Let $v \geqslant 6$ and $\lambda > 1$. Then an MPT(v,λ) can be embedded in a TS(u,λ) if and only if $\lambda(u-1) \equiv 0 \pmod 2$, $\lambda u(u-1) \equiv 0 \pmod 6$ and $u \geqslant 2v+1$.*

2. Basic construction techniques

Let $A_1 + A_2 + \cdots + A_n$ denote the union of multisets $A_1, A_2, \ldots, A_{n-1}$ and A_n (so if e occurs k_i times in A_i for any $1 \leqslant i \leqslant n$, then it occurs $k_1 + k_2 + \cdots + k_n$ times in $A_1 + A_2 + \cdots + A_n$). If $A_1 = A_2 = \cdots = A_n = A$, then let nA denote $A_1 + A_2 + \cdots + A_n$. Let λK_n denote the multigraph on n vertices in which each pair of vertices is joined by exactly λ edges. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a multigraph G respectively. Given two multigraphs G and H, the union $G \cup H$ is the graph with $V(G\cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) + E(H)$. If $V(G) \cap V(H) = \emptyset$, then the join $G \vee H$ is the graph with $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) + E(H) + \{\{x,y\} | x \in V(G), y \in V(H)\}$. If G is a multigraph, then let λG denote the graph with $V(\lambda G) = V(G)$ and $E(\lambda G) = \lambda E(G)$. Let $\bar{G}$ denote the complement of the graph G. We refer to [4] for an overview on graph theory.

Construction 2.1. *Suppose (X,A) is an* MPT(v,λ_1) *with leave $L(v,\lambda_1)$ and (X,B) is an* MPT(v,λ_2) *with leave $L(v,\lambda_2)$. Let $u > v$, and u be both λ_1-admissible and λ_2-admissible. If the* MPT(v,λ_1) *can be embedded in a* TS(u,λ_1), *the* MPT(v,λ_2) *can be embedded in a* TS(u,λ_2) *and $L(v,\lambda_1) \cup L(v,\lambda_2)$ is a $L(v,\lambda_1+\lambda_2)$, then an* MPT$(v,\lambda_1+\lambda_2)$ *with leave $L(v,\lambda_1+\lambda_2)$ can be embedded in a* TS$(u,\lambda_1+\lambda_2)$.

Proof. Let $V(\bar{K}_v) = X$, $V(K_{u-v}) = Z$ and $X \cap Z = \emptyset$. We recall that embedding an MPT$(v,\lambda_1+\lambda_2)$ in a TS$(u,\lambda_1+\lambda_2)$ is equivalent to partitioning the set of edges in $(\lambda_1+\lambda_2)(\bar{K}_v \vee K_{u-v}) \cup L(v,\lambda_1+\lambda_2)$ into triples. The union of the graph $\lambda_1(\bar{K}_v \vee K_{u-v}) \cup L(v,\lambda_1)$ and the graph $\lambda_2(\bar{K}_v \vee K_{u-v}) \cup L(v,\lambda_2)$ is exactly the graph $(\lambda_1+\lambda_2)(\bar{K}_v \vee K_{u-v}) \cup L(v,\lambda_1+\lambda_2)$ since $L(v,\lambda_1) \cup L(v,\lambda_2) = L(v,\lambda_1+\lambda_2)$. Since each of the set of edges in $\lambda_1(\bar{K}_v + K_{u-v}) \cup L(v,\lambda_1)$ and $\lambda_2(\bar{K}_v + K_{u-v}) \cup L(v,\lambda_2)$ respectively can be partitioned into triples, the conclusion follows. □

Theorem 2.2. *Let $u > v \geqslant 6$, $v \equiv 0, 4 \pmod 6$, $u \equiv 5 \pmod 6$. Then an MPT$(v,3)$ can be embedded in a TS$(u,3)$ if $u \geqslant 2v+1$.*

Proof. We prove the theorem for the case $v \equiv 0 \pmod 6$, the case $v \equiv 4 \pmod 6$ can be dealt with in a similar way. Let X be a v-set, $Y = X \cup Z_{u-v}$ and $X \cap Z_{u-v} = \emptyset$. By Mendelsohn and Rosa [10] there exists an MPT$(u,1)$ (Y,B_1) with leave $\{\{0,1\},\{1,2\},\{2,3\},\{0,3\}\}$ containing an MPT$(v,1)$ (X,A_1) as a subsystem. By Milici et al. [12] there exists an MPT$(u,2)$ (Y,B_2) with leave $\{\{0,2\},\{0,2\}\}$ containing a TS$(v,2)$ (X,A_2) as a subsystem. Let $A = A_1 + A_2$ and $B = B_1 + B_2 + \{\{0,1,2\},\{0,2,3\}\}$. Then (X,A) is an MPT$(v,3)$ and (Y,B) is a *TS*$(u,3)$ containing (X,A) as a subsystem. This completes the proof. □

By using Theorems 1.1, 1.4, 1.5, 2.2 and Construction 2.1, we have the following theorem:

Theorem 2.3. *Let $v \geqslant 6$, $\lambda > 1$, and let u be λ-admissible. If $u \geqslant 2v+1$, then any MPT(v,λ) can be embedded in a TS(u,λ) with the following possible exceptions*:

(1) $v \equiv 2 \pmod 6$, $\lambda \equiv 3 \pmod 6$, *and the leave of MPT(v,λ) is $1F_5$, $1FH$ or $1FYY$, and*
(2) $v \equiv 2 \pmod 6$ *and* $\lambda \equiv 5 \pmod 6$.

In order to eliminate all of these possible exceptions, by Construction 2.1 and Theorem 2.3, we only need to embed an MPT(v,λ) in a *TS*(u,λ) for $v \equiv 2 \pmod 6$, $\lambda = 3$ or 5, and all admissible $u \geqslant 2v+1$.

Now let X be a v-set and $v \geqslant 2$. A collection F of 2-subsets (called pairs) of X is called a partial λ-factor if each vertex of X is contained in at most λ pairs of F, and F is called a λ-factor if each vertex of X is contained in exactly λ pairs of F. If F is a partial λ-factor and $a \in X$, then let $a * F$ denote the multiset $\{\{a,x,y\} | \{x,y\} \in F\}$.

Construction 2.4. *Let $u > v \geqslant 6$, $\lambda = 2\lambda_0 + 1 \geqslant 3$, u be λ-admissible, $V(K_v) = X = \{a_i | 1 \leqslant i \leqslant v\}$, $V(K_{u-v}) = Z$, $X \cap Z = \phi$ and $Y = X \cup Z$. Let (X, A) be an* MPT(v, λ) *with leave $L(v, \lambda)$. If we can remove enough triples (denoted the collection of all these triples by H) from $G = \lambda(\bar{K}_v \vee K_{u-v}) \cup L(v, \lambda)$ such that $G - H$ satisfies:*

(1) *each edge of $L(v, \lambda)$ has been removed;*
(2) *there are only $2\lambda_0$ edges joining x and z for each $x \in X$ and $z \in Z$; and*
(3) *the subgraph of $G - H$ induced by Z is a $2\lambda_0 v$-regular subgraph, then an* MPT(v, λ) *with leave $L(v, \lambda)$ can be embedded in a* TS(u, λ).

Proof. As $2\lambda_0 v$ is even, by Petersen [3] the subgraph of $G - H$ induced by Z has a 2-factorization $C = \{F_{ij} | 1 \leqslant i \leqslant v, 1 \leqslant j \leqslant \lambda_0\}$. For any $1 \leqslant i \leqslant v$, let $B_i = a_i * F_{i1} + a_i * F_{i2} + \cdots + a_i * F_{i\lambda_0}$. Let $B = A + H + B_1 + B_2 + \cdots + B_v$. Then (Y, B) is a $TS(u, \lambda)$ containing (X, A) (MPT(v, λ)) as a subsystem. This completes the proof. □

3. 1-factors and partial 1-factors in complete graphs

Lemma 3.1. *Let $n = 6t + s + 4$, $s = 0, 2$ or 4, $n \geqslant 14$, and $V(K_n) = \{a, b, c, d, \infty\} \cup Z_{6t+s-1}$. Then K_n contains a subgraph whose edge set can be partitioned into 7 partial 1-factors each missing exactly the four vertices of $\{a, b, c, d\}$ and $6t + s - 8$ 1-factors each containing no edges of the complete graph K_4 on $\{a, b, c, d\}$.*

Proof. The seven partial 1-factors are:

$$F_i^s = \{\{i + j, i - j\} | j \in Z_{3t+s/2} \setminus \{0\}\} \cup \{\{i, \infty\}\},$$
$$6t + s - 8 \leqslant i \leqslant 6t + s - 2.$$

Let $m = 1 - (-1)^t$. For $0 \leqslant i \leqslant 3t + s/2 - 5$, let

$$\begin{aligned} F_{2i}^s = {} & \{\{a, 3t + 2i + s/2 - 2\}, \{b, 3t + 2i + s/2 + 1\}, \{c, 3t + 2i + s/2\}\} \\ & \cup \{\{3t + 2i + 2j + s/2 + 2, 3t + 2i - 2j + s/2 - 3\} | \\ & \quad 0 \leqslant j \leqslant (6t + s - m - 8)/4\} \\ & \cup \{\{3t + 2i + 2j + s/2 + 3, 3t + 2i - 2j + s/2 - 4\} | \\ & \quad 0 \leqslant j \leqslant (6t + s + m - 12)/4\} \\ & \cup \{\{d, 3t + 2i + s/2 - 1\}, \{2i, \infty\}\}, \text{ and} \end{aligned}$$

$$\begin{aligned} F_{2i+1}^s = {} & \{\{a, 3t + 2i + s/2 + 1\}, \{b, 3t + 2i + s/2\}, \{c, 3t + 2i + s/2 - 1\}\} \\ & \cup \{\{3t + 2i + 2j + s/2 + 4, 3t + 2i - 2j + s/2 - 3\} | \\ & \quad 0 \leqslant j \leqslant (6t + s + m - 12)/4\} \\ & \cup \{\{3t + 2i + 2j + s/2 + 3, 3t + 2i - 2j + s/2 - 2\} | \\ & \quad 0 \leqslant j \leqslant (6t + s - m - 8)/4\} \\ & \cup \{\{d, 3t + 2i + s/2 + 2\}, \{2i + 1, \infty\}\}. \end{aligned}$$

Then $F_0^s, F_1^s, \ldots, F_{6t+s-9}^s$ are the desired $6t + s - 8$ 1-factors. This completes the proof. □

Since there are 6 edges in the complete graph K_4, the following lemma is obvious:

Lemma 3.2. *Let $n = 6h + s + 4$, $s = 0$ or 2, $n \geqslant 12$, and $V(K_n) = \{a, b, c, d, \infty\} \cup Z_{6h+s-1}$. Then K_n contains a subgraph whose edge set can be partitioned into $6h + s - 4$ 1-factors each containing no edges of the complete graph K_4 on $\{a, b, c, d\}$.*

Lemma 3.3. *Let $n = 6h + 9 \geqslant 15$ and $V(K_n) = \{a, b, c, d, e, \infty\} \cup Z_{6h+3}$. Then $2K_n$ contains a subgraph whose edge set can be partitioned into $6h + 9$ partial 1-factors $F_0, F_1, \ldots, F_{6h+7}$ and F_{6h+8} satisfying the following properties*:

(1) *for any $0 \leqslant i \leqslant 3h$, each of F_{2i} and F_{2i+1} misses exactly the vertex i and contains no edges of the complete graph K_5 on $\{a, b, c, d, e\}$*;
(2) *for any $6h + 2 \leqslant i \leqslant 6h + 8$, F_i misses exactly the five vertices of $\{a, b, c, d, e\}$.*

Proof. Set $t = h$ in Lemma 3.1, and let $F_{i+6} = F_i^4$ for any i where $6h - 4 \leqslant i \leqslant 6h + 2$. Now choose $4h$ 1-factors $F_0^4, F_1^4, \ldots, F_{4h-2}^4$ and F_{4h-1}^4 in the proof of Lemma 3.1. For any i where $0 \leqslant i \leqslant 2h - 1$, we replace the vertex i in F_{2i}^4 and F_{2i+1}^4 by e. Then we obtain $4h$ partial 1-factors (denoted by $F_0, F_1, \ldots, F_{4h-2}$ and F_{4h-1} correspondingly) in $2K_n$. It is easy to check that there exist no repeated edges in these $4h$ partial 1-factors. So we can again choose $2h + 2$ 1-factors $F_0^4, F_1^4, \ldots, F_{2h}^4$ and F_{2h+1}^4 in the proof of Lemma 3.1. Let $\sigma = (0, 2h)(1, 2h + 1) \ldots (h, 3h)$ be a permutation of $V(K_n)$. For any i where $2h \leqslant i \leqslant 3h$, let $k = i - 2h$ and set

$$F_{0i} = \sigma(F_{2k}^4) = \{\{\sigma(x), \sigma(y)\} \mid \{x, y\} \in F_{2k}^4\}, \text{ and}$$

$$F_{1i} = \sigma(F_{2k+1}^4) = \{\{\sigma(x), \sigma(y)\} \mid \{x, y\} \in F_{2k+1}^4\}.$$

If we replace the vertex i in F_{0i} and F_{1i} by e for any i where $2h \leqslant i \leqslant 3h$, then we obtain $2h+2$ new partial 1-factors (denoted by $F_{4h}, F_{4h+1}, \ldots, F_{6h}$ and F_{6h+1} correspondingly) in $2K_n$. It is also easy to check that there exist no repeated edges in all these $2h + 2$ partial 1-factors. Therefore, $F_0, F_1, \ldots, F_{6h+7}$ and F_{6h+8} are the desired $6h + 9$ partial 1-factors. This completes the proof. □

Similarly, the following lemma can also be proved, we omit the details here.

Lemma 3.4. *Let $n = 18h + 21 \geqslant 21$ and $V(K_n) = \{a, b, c, d, e, \infty\} \cup Z_{18h+15}$. Then K_n contains a subgraph whose edge set can be partitioned into $6h + 9$ partial 1-factors $F_0, F_1, \ldots, F_{6h+7}$ and F_{6h+8} satisfying properties (1) and (2) of Lemma 3.3.*

Lemma 3.5. *Let $n=6t+s+2$, $s=2$ or 4, $t \geqslant 1$ and $V(K_n)=\{a, b, \infty\} \cup Z_{6t+s-1}$. Then K_n contains a subgraph whose edge set can be partitioned into 5 partial 1-factors each missing exactly the two vertices of $\{a, b\}$ and $6t + s - 6$ 1-factors each containing no edge $\{a, b\}$.*

Proof. The five partial 1-factors are:

$$F_i^s = \{\{i + j, i - j\} | j \in Z_{3t+s/2} \setminus \{0\}\} \cup \{\{i, \infty\}\},$$

$$6t + s - 6 \leqslant i \leqslant 6t + s - 2.$$

Let $m = 1 - (-1)^t$. For $0 \leqslant i \leqslant 3t + s/2 - 4$, let

$$\begin{aligned} F_{2i}^s = {} & \{\{a, 3t + 2i + s/2 - 2\}, \{b, 3t + 2i + +s/2 + 1\}, \{2i, \infty\}\} \\ & \cup \{\{3t + 2i + 2j + s/2, 3t + 2i - 2j + s/2 - 1\} | \\ & \quad 0 \leqslant j \leqslant [6t + 2s - (-1)^{s/2} m - 8]/4\} \\ & \cup \{\{3t + 2i + 2j + s/2 + 3, 3t + 2i - 2j + s/2 - 4\} | \\ & \quad 0 \leqslant j \leqslant [6t + (-1)^{s/2} m - 8]/4\}, \text{ and} \end{aligned}$$

$$\begin{aligned} F_{2i+1}^s = {} & \{\{a, 3t + 2i + s/2 + 1\}, \{b, 3t + 2i + s/2\}, \{1 + 2i, \infty\}\} \\ & \cup \{\{3t + 2i + 2j + s/2 + 2, 3t + 2i - 2j + s/2 - 1\} | \\ & \quad 0 \leqslant j \leqslant [6t + (-1)^{s/2} m - 4]/4\} \\ & \cup \{\{3t + 2i + 2j + s/2 + 3, 3t + 2i - 2j + s/2 - 2\} | \\ & \quad 0 \leqslant j \leqslant [6t + 2s - (-1)^{s/2} m - 12]/4\}. \end{aligned}$$

Then $F_0^s, F_1^s, \ldots, F_{6t+s-7}^s$ are the desired $6t + s - 6$ 1-factors. This completes the proof. □

We also need the following theorem in Section 4.

Theorem 3.6 (Colbourn and Rosa [1]). *Let n be an odd integer and $n \neq 9$. Then for each 2-regular subgraph Z of K_n, $K_n - Z$ can be partitioned into triples if and only if $3|(n(n-1)/2 - |E(Z)|)$.*

4. The case $\lambda = 3$ and $v \equiv 2 \pmod 6$

For $n \equiv 1 \pmod 2$, as in [14], let $D(n,\lambda)$ be the multiset $\lambda\{d|1 \leqslant d \leqslant (n-1)/2\}$ with elements from Z_n. The elements of $D(n,\lambda)$ are called differences. We remark that we also use $n-d$ to represent the difference d.

Let $a,b,c \in D(n,\lambda)$, if $a+b+c \equiv 0 \pmod n$ or one is the sum of the others, say, $a+b \equiv c \pmod n$, then $D=(a,b,c)$ is called a difference triple, let (D) denote the set of triples $\{\{0,a,a+b\}+i|0 \leqslant i \leqslant n-1\}$ or $\{\{0,b,a+b\}+i|0 \leqslant i \leqslant n-1\}$, and we say that (D) is induced by the difference triple D. If $n \equiv 0 \pmod 3$, and $a=b=c=n/3$, then $\{\{0,n/3,2n/3\}+i|0 \leqslant i \leqslant n/3-1\}$ can form a 2-regular spanning subgraph of K_n on the vertex set Z_n. In this case, the difference $n/3$ is used once. Otherwise, the triple set induced by the difference triple (a,b,c) forms a 6-regular spanning subgraph of K_n on Z_n.

In this section, let $v \equiv 2 \pmod 6$, $v \geqslant 8$, $V(K_v)=V(\bar{K}_v)=X=\{a_i|0 \leqslant i \leqslant v-1\}$, and let (X,A) be an MPT$(v,3)$ with leave $L(v,3)=1F_5$, $1FH$ or $1FYY$. Let $u>v$, $Y=X\cup Z_{u-v}$, $X\cap Z_{u-v}=\emptyset$ and $V(K_{u-v})=Z_{u-v}$. Set

$$1F_5=\{\{a_0,a_i\}|1 \leqslant i \leqslant 5\} \cup \{\{a_{2i},a_{2i+1}\}|3 \leqslant i \leqslant v/2-1\};$$

$$1FH=\{\{a_0,a_1\},\{a_1,a_2\},\{a_1,a_3\},\{a_3,a_4\},\{a_3,a_5\}\} \cup \{\{a_{2i},a_{2i+1}\}|3 \leqslant i \leqslant v/2-1\};$$

$$\begin{aligned}1FYY=\{&\{a_0,a_1\},\{a_0,a_2\},\{a_0,a_6\},\{a_4,a_5\},\{a_3,a_4\},\{a_4,a_7\}\}\\ &\cup \{\{a_{2i},a_{2i+1}\}|4 \leqslant i \leqslant v/2-1, v>8\};\end{aligned}$$

$$\begin{aligned}A_0=\{&\{a_1,1,4\},\{a_1,2,3\},\{a_2,0,2\},\{a_2,3,4\},\{a_3,1,3\},\{a_3,0,4\},\\ &\{a_4,0,1\},\{a_4,2,4\},\{a_5,0,3\},\{a_5,1,2\}\} \cup \{\{a_0,a_i,i-1\}|1 \leqslant i \leqslant 5\};\end{aligned}$$

$$\begin{aligned}A_1=\{&\{a_0,a_1,0\},\{a_1,a_2,1\},\{a_1,a_3,2\},\{a_3,a_4,3\},\{a_3,a_5,4\},\{a_1,3,4\},\\ &\{a_2,2,4\},\{a_2,0,3\},\{a_3,0,1\},\{a_4,1,2\},\{a_4,0,4\},\{a_5,1,3\},\{a_5,0,2\},\\ &\{a_0,1,4\},\{a_0,2,3\}\}, \text{ and}\end{aligned}$$

$$\begin{aligned}A_2=\{&\{a_0,a_1,0\},\{a_0,a_2,1\},\{a_4,a_5,2\},\{a_3,a_4,3\},\{a_2,0,3\},\{a_3,1,4\},\{a_3,0,2\},\\ &\{a_2,2,4\},\{a_1,1,2\},\{a_1,3,4\},\{a_5,0,4\},\{a_5,1,3\},\{a_0,2,3\},\{a_4,0,1\}\}.\end{aligned}$$

Lemma 4.1. *Let $v \equiv 2 \pmod 6$, $v \geqslant 8$ and $u \equiv 1 \pmod 6$. If $u \geqslant 2v+1$, then an MPT$(v,3)$ with leave $1F_5$, $1FH$ or $1FYY$ can be embedded in a TS$(u,3)$.*

Proof. Write $v=6h+8$ and $u-v=6t+5$. By Theorem 1.2, we can suppose $2v+1 \leqslant u \leqslant 4v$, so $1 \leqslant h+1 \leqslant t \leqslant 3h+3$.

Case 1: $h=0$ and $t=1$. We take 6 1-factors F_0, F_1, F_2, F_3, F_4 and F_5 in $2K_6$ on $\{5,6,7,8,9,10\}$ such that each of the edges $\{5,6\}$ and $\{9,10\}$ is not contained in any of the above 1-factors. Let

$$\begin{aligned}B_1=\{&\{4,a_6,a_7\},\{0,5,a_6\},\{1,6,a_6\},\{2,7,a_6\},\{3,8,a_6\},\{9,10,a_6\},\{0,7,a_7\},\\ &\{1,8,a_7\},\{2,9,a_7\},\{3,10,a_7\},\{5,6,a_7\},\{0,1,4\},\{2,3,4\}\},\end{aligned}$$

$B_2=\{\{0,1,3\}+i|0 \leqslant i \leqslant 10\}$, $B_3=\bigcup_{i=0}^{5}(a_i * F_i)$, and $B_4=(B_1 \setminus \{\{a_6,a_7,4\}\}) \cup \{\{4,a_0,a_6\},\{4,a_4,a_7\}\}$. Set

$$H=\begin{cases} A_0+B_1+B_2+B_3, & \text{if } L(v,3)=1F_5,\\ A_1+B_1+B_2+B_3, & \text{if } L(v,3)=1FH, \text{ and}\\ A_2+B_2+B_3+B_4, & \text{if } L(v,3)=1FYY.\end{cases}$$

Case 2: $t = h + 1 \geqslant 2$. By Lemmas 3.1 and 3.2, $2K_{6h+11}$ contains 7 partial 1-factors F_0, F_1, F_2, F_3, F_4, F_5 and F each missing exactly the five vertices of $\{0,1,2,3,4\}$ and $6h+2$ partial 1-factors $F_6, F_7, \ldots, F_{6h+6}$ and F_{6h+7} each missing exactly the vertex 4 and containing no edges of K_4 on $\{0,1,2,3\}$. Suppose $F = \{\{2i-1, 2i\} | 3 \leqslant i \leqslant 3h+5\}$. Let $B_1 = \{\{4, a_{2i}, a_{2i+1}\} | 3 \leqslant i \leqslant 3h+3\}$, $B_2 = (\bigcup_{i=0}^{6h+7} a_i * F_i) \cup \{\{4, 2i-1, 2i\} | 3 \leqslant i \leqslant 3h+2\}$ and $\sigma = (4h+5, 6h+5)$ be a transposition of Z_{6h+11}. Now in the third copy of K_{6h+11} we take the difference triple $(2h, 2h+1, 2h+10)$. Let $B_0 = \{\{0, 2h, 4h+1\} + i | 0 \leqslant i \leqslant 4, 2h+5 \leqslant i \leqslant 6h+10\}$, $B_3 = \{\{4, a_{2i}, a_{2i+1}\} \mid 4 \leqslant i \leqslant 3h+3\} \cup \{\{4, a_0, a_6\}, \{4, a_4, a_7\}\}$, and $B_4 = \{\{0,1,4\}, \{2,3,4\}\}$. Set

$$H = \begin{cases} A_0 + B_1 + B_2 + B_4 + \sigma(B_0), & \text{if } L(v,3) = 1F_5, \\ A_1 + B_1 + B_2 + B_4 + \sigma(B_0), & \text{if } L(v,3) = 1FH, \text{ and} \\ A_2 + B_2 + B_3 + B_4 + \sigma(B_0), & \text{if } L(v,3) = 1FYY. \end{cases}$$

Case 3: $2 \leqslant h+2 \leqslant t \leqslant 3h+2$. By Lemma 3.1, K_{u-v} contains 7 partial 1-factors F_0, F_1, F_2, F_3, F_4, F_5 and F each missing exactly the five vertices of $\{0,1,2,3,4\}$ and $6h+2$ partial 1-factors $F_6, F_7, \ldots, F_{6h+6}$ and F_{6h+7} each missing exactly the vertex 4 and containing no edges of K_4 on $\{0,1,2,3\}$. Suppose $F = \{\{2i-1, 2i\} | 3 \leqslant i \leqslant 3t+2\}$. Let B_1, B_2 and B_3 be the same as defined in Case 2.

Choose differences of $D(6t+5, 2)$ to form the following collection T of $2t+1$ difference triples:

$$(1, 2t+2, 2t+3),\ (2t, 2t+2, 2t+3),\ (2, 2t-3, 2t-1),$$

$$(3, 2t-2, 2t+1),\ (2t, 2t+1, 2t+4),$$

$$(2i, 3t-i+2, 3t-i+3) : 1 \leqslant i \leqslant t-2,\ t \geqslant 3, \quad \text{and}$$

$$(2i-1, t+i+1, t-i+2) : 1 \leqslant i \leqslant t-2,\ t \geqslant 3.$$

From $T \setminus \{(1, 2t+2, 2t+3), (2t, 2t+2, 2t+3)\}$, choose $3t-3h-3$ difference triples $D_1, D_2, \ldots, D_{3t-3h-4}$ and $D_{3t-3h-3}$. Let $B_4 = (D_1) + (D_2) + \cdots + (D_{3t-3h-3})$, $B_5 = \{\{0, 2t, 4t+2\} + i | 0 \leqslant i \leqslant 4, 2h+5 \leqslant i \leqslant 6t+4\} \cup \{\{0, 1, 2t+3\}, \{1, 2, 2t+4\}\}$, the vertex set $V_1 = \{0, 1, 2, 2t+3, 2t+4\}$, and the vertex set $V_2 = \{i | 5 \leqslant i \leqslant 2h+4,\ 2t+5 \leqslant i \leqslant 2t+2h+4,\ 4t+7 \leqslant i \leqslant 4t+2h+6\}$. Clearly,[1] $V_1 \cap V_2 = \phi$, so there exists a permutation σ of Z_{6t+5} such that $\sigma(0) = 0$, $\sigma(1) = 4$, $\sigma(2) = 2$, $\sigma(2t+3) = 1$, $\sigma(2t+4) = 3$ and $\sigma(V_2) = \{i | 5 \leqslant i \leqslant 6h+4\}$. Set

$$H = \begin{cases} B_1 + B_2 + \sigma(B_4) + \sigma(B_5) + A_0, & \text{if } L(v,3) = 1F_5, \\ B_1 + B_2 + \sigma(B_4) + \sigma(B_5) + A_1, & \text{if } L(v,3) = 1FH, \text{ and} \\ B_2 + B_3 + \sigma(B_4) + \sigma(B_5) + A_2, & \text{if } L(v,3) = 1FYY. \end{cases}$$

Case 4: $t = 3h+3 \geqslant 3$. Let the partial 1-factors $F_0, F_1, \ldots, F_{6h+7}$ and F in K_{18h+23} be similar to Case 3, B_1 and B_3 be the same as defined in Case 2, and $B_2 = (\bigcup_{i=0}^{6h+7} a_i * F_i) \cup \{\{4, 2i-1, 2i\} | 3 \leqslant i \leqslant 3h+4\}$. Let C_1 denote any $(6h+4)$-cycle on the vertex set $\{i | 5 \leqslant i \leqslant 6h+8\}$, and C_2 any $(18h+19)$-cycle on the vertex set $\{i | 4 \leqslant i \leqslant 18h+22\}$. By Theorem 3.6, the graph $2K_{18h+23} - (C_1 \cup C_2)$ can be partitioned into a collection B_4 of triples. Set

$$H = \begin{cases} B_1 + B_2 + B_4 + A_0, & \text{if } L(v,3) = 1F_5, \\ B_1 + B_2 + B_4 + A_1, & \text{if } L(v,3) = 1FH, \text{ and} \\ B_2 + B_3 + B_4 + A_2, & \text{if } L(v,3) = 1FYY. \end{cases}$$

It can be checked that in each of the above cases, $3(\bar{K}_v \vee K_{u-v}) \cup L(v,3) - H$ satisfies conditions (1)–(3) of Construction 2.4. The conclusion follows. $\square$

Lemma 4.2. *Let* $v \equiv 2 \pmod 6$, $v \geqslant 8$ *and* $u \equiv 3 \pmod 6$. *If* $u \geqslant 2v+1$, *then an* $MPT(v,3)$ *with leave* $1F_5$, $1FH$ *or* $1FYY$ *can be embedded in a* $TS(u,3)$.

Proof. Write $v = 6h+8$ and $u - v = 6t+7$. Suppose $1 \leqslant h+1 \leqslant t \leqslant 3h+2$.

[1] **Remark**. If $h = 0$, then both of the subset $\{\{4, 2i-1, 2i\} | 3 \leqslant i \leqslant 3h+2\}$ of B_2 and the subset $\{\{4, a_{2i}, a_{2i+1}\} | 4 \leqslant i \leqslant 3h+3\}$ of B_3 are ϕ, so is the vertex set V_2. Also, for the following cases in this paper we have similar results.

Case 1: $t=h+1 \geqslant 1$. Let partial 1-factors $F_0, F_1, \ldots, F_{6h+7}$ and F in $2K_{6h+13}$ be similar to Case 2 in the proof of Lemma 4.1. If $h \geqslant 1$, then we choose two difference triples, $(2h, 2h+1, 2h+12)$ and $(1, 2h+10, 2h+11)$, from difference set $D(6h+13, 1)$. Let the vertex set $V=\{i|0 \leqslant i \leqslant 2h-1, 2h \leqslant i \leqslant 4h-1, 4h+1 \leqslant i \leqslant 6h\}$. Then there exists a permutation σ of Z_{6h+13} such that $\sigma(V)=\{i|5 \leqslant i \leqslant 6h+4\}$. Let B_1, B_2 and B_3 be the same as defined in Case 2 in the proof of Lemma 4.1, $B_5=\{\{0,1,4\},\{2,3,4\}\}$, $B_6=\{\{0,1,4h+3\}+i|0 \leqslant i \leqslant 6h+12\} \cup \{\{0,2h,4h+1\}+i|2h \leqslant i \leqslant 6h+12\}$,

$$B_4 = \begin{cases} \sigma(B_6), & \text{if } h \geqslant 1, \\ \{\{0,1,4\}+i|0 \leqslant i \leqslant 12\} \cup \{\{0,2,7\}+i|0 \leqslant i \leqslant 12\}, & \text{if } h=0, \text{ and} \end{cases}$$

$$H = \begin{cases} B_1+B_2+B_4+B_5+A_0, & \text{if } L(v,3)=1F_5, \\ B_1+B_2+B_4+B_5+A_1, & \text{if } L(v,3)=1FH, \text{ and} \\ B_2+B_3+B_4+B_5+A_2, & \text{if } L(v,3)=1FYY. \end{cases}$$

Case 2: $1 \leqslant h+1 \leqslant t \leqslant 3h+1$. Let the partial 1-factors $F_0, F_1, \ldots, F_{6h+7}$ and F in K_{u-v} be similar to Case 3 in the proof of Lemma 4.1. Choose differences of $D(6t+7,2)$ to form the following collection T of difference triples:

$$(2i, 3t-i+3, 3t-i+4) : 1 \leqslant i \leqslant t, \text{ and}$$

$$(2i-1, t+i+1, t-i+2) : 1 \leqslant i \leqslant t+1.$$

From $T \setminus \{(2t-2, 2t+4, 2t+5), (1, 2t+1, 2t+2)\}$, choose $3t-3h-2$ difference triples $D_1, D_2, \ldots, D_{3t-3h-3}$ and $D_{3t-3h-2}$. Let $B_4=(D_1)+(D_2)+\cdots+(D_{3t-3h-2})$, B_1, B_2 and B_3 be the same as defined in Case 2 in the proof of Lemma 4.1, $B_5=\{\{4t-1,4t,6t+1\},\{4t,4t+1,6t+2\}\} \cup \{\{0,2t-2,4t+2\}+i|2h \leqslant i \leqslant 6t+6\}$, and the vertex set $V=\{i|0 \leqslant i \leqslant 2h-1, 2t-2 \leqslant i \leqslant 2t+2h-3, 4t+2 \leqslant i \leqslant 4t+2h+1\}$. Clearly, there exists a permutation σ of Z_{6t+7} such that $\sigma(4t-1)=0$, $\sigma(4t)=4$, $\sigma(6t+1)=1$, $\sigma(4t+1)=2$, $\sigma(6t+2)=3$ and $\sigma(V)=\{i|5 \leqslant i \leqslant 6h+4\}$. Set

$$H = \begin{cases} B_1+B_2+\sigma(B_4)+\sigma(B_5)+A_0, & \text{if } L(v,3)=1F_5, \\ B_1+B_2+\sigma(B_4)+\sigma(B_5)+A_1, & \text{if } L(v,3)=1FH, \text{ and} \\ B_2+B_3+\sigma(B_4)+\sigma(B_5)+A_2, & \text{if } L(v,3)=1FYY \end{cases}$$

Case 3: $t=3h+2 \geqslant 2$. Let the partial 1-factors $F_0, F_1, \ldots, F_{6h+7}$ and F in K_{18h+19} be similar to Case 3 in the proof of Lemma 4.1. Choose differences of $D(18h+19,2)$ to form the following collection T of difference triples:

$$2\{(3h+3, 3h+4, 6h+7),$$
$$(3i+1, 4h-i+5, 4h+2i+6) : 0 \leqslant i \leqslant h,$$
$$(3i+3, 6h-2i+5, 6h+i+8) : 0 \leqslant i \leqslant h-1, \ h \geqslant 1,$$
$$(3i+2, 8h-i+8, 8h+2i+10) : 0 \leqslant i \leqslant h\}.$$

From $T \setminus \{(3h+3, 3h+4, 6h+7), (2, 8h+8, 8h+10)\}$, choose $6h+4$ difference triples $D_1, D_2, \ldots, D_{6h+3}$ and D_{6h+4}. Let B_1, B_2 and B_3 be the same as defined in Case 2 in the proof of Lemma 4.1, $B_4=(D_1)+(D_2)+\cdots+(D_{6h+4})$, $B_5=\{\{2h+1,2h+3,10h+11\},\{2h+3,2h+5,10h+13\}\} \cup \{\{0,3h+3,6h+7\}+i|2h \leqslant i \leqslant 18h+18\}$, and the vertex set $V=\{i|0 \leqslant i \leqslant 2h-1, 3h+3 \leqslant i \leqslant 5h+2, 6h+7 \leqslant i \leqslant 8h+6\}$. Clearly, there exists a permutation σ of Z_{18h+19} such that $\sigma(2h+1)=0$, $\sigma(2h+3)=4$, $\sigma(10h+11)=1$, $\sigma(2h+5)=2$, $\sigma(10h+13)=3$ and $\sigma(V)=\{i|5 \leqslant i \leqslant 6h+4\}$. Set

$$H = \begin{cases} B_1+B_2+\sigma(B_4)+\sigma(B_5)+A_0, & \text{if } L(v,3)=1F_5, \\ B_1+B_2+\sigma(B_4)+\sigma(B_5)+A_1, & \text{if } L(v,3)=1FH, \text{ and} \\ B_2+B_3+\sigma(B_4)+\sigma(B_5)+A_2, & \text{if } L(v,3)=1FYY. \end{cases}$$

It can be checked that in each of the above cases, $3(\bar{K}_v \vee K_{u-v}) \cup L(v,3) - H$ satisfies conditions (1)–(3) of Construction 2.4. The conclusion follows. $\square$

Lemma 4.3. *Let $v \equiv 2 \pmod 6$, $v \geqslant 8$ and $u \equiv 5 \pmod 6$. If $u \geqslant 2v+1$, then an MPT$(v,3)$ with leave $1F_5$, $1FH$ or $1FYY$ can be embedded in a TS$(u,3)$.*

Proof. Write $v = 6h+8$ and $u-v = 6t+9$. Suppose $0 \leqslant h \leqslant t \leqslant 3h+2$.

Case 1: $t = h = 0$. Let $\{F_0, F_1, F_2\}$ be a 1-factorization of K_4 on $\{5,6,7,8\}$, $B_1 = (\bigcup_{i=0}^{2} a_i * F_i) \cup (\bigcup_{i=3}^{5} a_i * F_{i-3}) \cup \{\{0,1,4\},\{2,3,4\}\}$, $B_2 = \{\{a_j, i, i+5\} | 0 \leqslant i \leqslant 3; j = 6,7\} \cup \{\{4, a_6, a_7\}\}$, and $B_3 = \{\{a_j, i, i+5\} | 0 \leqslant i \leqslant 3; j = 6,7\} \cup \{\{4, a_0, a_6\}, \{4, a_4, a_7\}\}$. Set

$$H = \begin{cases} B_1 + B_2 + A_0, & \text{if } L(v,3) = 1F_5, \\ B_1 + B_2 + A_1, & \text{if } L(v,3) = 1FH, \text{ and} \\ B_1 + B_3 + A_2, & \text{if } L(v,3) = 1FYY. \end{cases}$$

Case 2: $t = h \geqslant 1$ and $L(v,3) = 1F_5$ or $1FH$. By Lemma 3.3, $2K_{6h+9}$ contains 6 partial 1-factors F_0, F_1, F_2, F_3, F_4 and F_5 each missing exactly the five vertices of $\{0,1,2,3,4\}$ and $6h+2$ partial 1-factors $F_6, F_7, \ldots, F_{6h+6}$ and F_{6h+7} such that each of F_{2i} and F_{2i+1} misses exactly the vertex $i+2$ for each i where $3 \leqslant i \leqslant 3h+3$. Let $\sigma = (3h+6, 2h+4)$ be a transposition of Z_{6h+9}, $B_1 = (\bigcup_{i=0}^{6h+7} a_i * F_i) \cup \{\{i+2, a_{2i}, a_{2i+1}\} | 3 \leqslant i \leqslant 3h+3\}$, and $B_2 = \{\{0, h+2, 2h+5\} + i | 0 \leqslant i \leqslant h+1\}$. Set

$$H = \begin{cases} B_1 + \sigma(B_2) + A_0, & \text{if } L(v,3) = 1F_5, \text{ and} \\ B_1 + \sigma(B_2) + A_1, & \text{if } L(v,3) = 1FH. \end{cases}$$

Case 3: $t = h \geqslant 1$ and $L(v,3) = 1FYY$. By Lemmas 3.1 and 3.3, $2K_{6h+9}$ contains 6 partial 1-factors F_0, F_1, F_2, F_3, F_4 and F_5 each missing exactly the five vertices of $\{0,1,2,3,4\}$ and 2 partial 1-factors F_6 and F_7 each missing exactly the vertex 4 and containing no edges of K_4 on $\{0,1,2,3\}$ and $6h$ partial 1-factors $F_8, F_9, \ldots, F_{6h+6}$ and F_{6h+7} such that each of F_{2i} and F_{2i+1} misses the vertex $i+1$ for each i where $4 \leqslant i \leqslant 3h+3$. Let $\sigma = (2h+5, 3h+5)$ be a transposition of Z_{6h+9}, $B_1 = (\bigcup_{i=0}^{6h+7} a_i * F_i) \cup \{\{0,1,4\},\{2,3,4\}\}$, $B_2 = \{\{i+1, a_{2i}, a_{2i+1}\} | 4 \leqslant i \leqslant 3h+3\} \cup \{\{4, a_0, a_6\}, \{4, a_4, a_7\}\}$, and $B_4 = \{\{0, h, 2h+1\} + i | 5 \leqslant i \leqslant h+4\}$. Set

$$H = B_1 \cup B_2 \cup \sigma(B_4) \cup A_3.$$

Case 4: $1 \leqslant h+1 \leqslant t \leqslant 3h+1$. Let the partial 1-factors $F_0, F_1, \ldots, F_{6h+7}$ and F in K_{u-v} be similar to Case 3 in the proof of Lemma 4.1. Choose differences of $D(6t+9, 2)$ to form the following collection T of difference triples:

$$(2i, 3t-i+4, 3t-i+5) : 1 \leqslant i \leqslant t+1, \text{ and}$$

$$(2i-1, t+i+1, t-i+2) : 1 \leqslant i \leqslant t+1.$$

From $T \setminus \{(2t-2, 2t+5, 2t+6), (1, 2t+1, 2t+2)\}$, choose $3t-3h-1$ difference triples $D_1, D_2, \ldots, D_{3t-3h-2}$ and $D_{3t-3h-1}$. Let B_1, B_2 and B_3 be the same as defined in Case 2 in the proof of Lemma 4.1, $B_4 = (D_1) + (D_2) + \cdots + (D_{3t-3h-1})$, $B_5 = \{\{4t, 4t+1, 6t+2\}, \{4t+1, 4t+2, 6t+3\}\} \cup \{\{0, 2t-2, 4t+3\} + i | 2h \leqslant i \leqslant 6t+8\}$, and the vertex set $V = \{i | 0 \leqslant i \leqslant 2h-1, 2t-2 \leqslant i \leqslant 2t+2h-3, 4t+3 \leqslant i \leqslant 4t+2h+2\}$. Clearly, there exists a permutation σ of Z_{6t+9} such that $\sigma(4t) = 0$, $\sigma(4t+1) = 4$, $\sigma(4t+2) = 2$, $\sigma(6t+2) = 1$, $\sigma(6t+3) = 3$ and $\sigma(V) = \{i | 5 \leqslant i \leqslant 6h+4\}$. Set

$$H = \begin{cases} B_1 + B_2 + \sigma(B_4) + \sigma(B_5) + A_0, & \text{if } L(v,3) = 1F_5, \\ B_1 + B_2 + \sigma(B_4) + \sigma(B_5) + A_1, & \text{if } L(v,3) = 1FH, \text{ and} \\ B_2 + B_3 + \sigma(B_4) + \sigma(B_5) + A_2, & \text{if } L(v,3) = 1FYY. \end{cases}$$

Case 5: $t = 3h+2 \geqslant 2$ and $L(v,3) = 1F_5$ or $1FH$. By Lemma 3.4, K_{18h+21} contains 6 partial 1-factors F_0, F_1, F_2, F_3, F_4 and F_5 each missing exactly the five vertices of $\{0,1,2,3,4\}$ and $6h+2$ partial 1-factors $F_6, F_7, \ldots, F_{6h+6}$ and F_{6h+7} such that each of F_{2i} and F_{2i+1} misses exactly the vertex $i+2$ and contains no edges of K_5 on $\{0,1,2,3,4\}$ for each i where $3 \leqslant i \leqslant 3h+3$. Let C_1 denote a $(15h+15)$-cycle on the vertex set $\{i | 3h+6 \leqslant i \leqslant 18h+20\}$, C_2 a $(18h+21)$-cycle on Z_{18h+21}. By Theorem 3.6, the graph $2K_{18h+21} - (C_1 \cup C_2)$ can be partitioned into a collection B_2 of triples. Let $B_1 = (\bigcup_{i=0}^{6h+7} a_i * F_i) \cup \{\{i+2, a_{2i}, a_{2i+1}\} | 3 \leqslant i \leqslant 3h+3\}$. Set

$$H = \begin{cases} B_1 + B_2 + A_0, & \text{if } L(v,3) = 1F_5, \text{ and} \\ B_1 + B_2 + A_1, & \text{if } L(v,3) = 1FH. \end{cases}$$

Case 6: $t = 3h + 2 \geqslant 2$ and $L(v,3) = 1FYY$. By Lemmas 3.1 and 3.4, K_{18h+21} contains $6h + 8$ partial 1-factors $F_0, F_1, \ldots, F_{6h+6}$ and F_{6h+7} which satisfy the following properties:

(1) for any $0 \leqslant i \leqslant 5$, F_i misses exactly the five vertices of $\{0,1,2,3,4\}$;
(2) for any $3 \leqslant i \leqslant 3h + 2$, each of F_{2i} and F_{2i+1} misses exactly the vertex $i + 1$ and contains no edges of K_5 on $\{0,1,2,3,4\}$;
(3) each of F_{6h+6} and F_{6h+7} misses exactly the vertex 4; and
(4) for any $0 \leqslant i \leqslant 6h + 7$, F_i contains no edges of $\{\{5,6\},\{4,5\},\{4,6\}\}$.

Let $B_1 = (\bigcup_{i=0}^{5} a_i * F_i) \cup (\bigcup_{i=6}^{6h+5} a_{i+2} * F_i) \cup (a_6 * F_{6h+6}) \cup (a_7 * F_{6h+7}) \cup \{\{i+1, a_{2i}, a_{2i+1}\} | 4 \leqslant i \leqslant 3h+3\} \cup \{\{4, a_0, a_6\}, \{4, a_4, a_7\}, \{4,5,6\}\}$, and let C_1 denote a $(15h+18)$-cycle on the vertex set $\{i | 3h+5 \leqslant i \leqslant 18h+20\} \cup \{5,6\}$, C_2 a $(18h+21)$-cycle on Z_{18h+21}. By Theorem 3.6, the graph $2K_{u-v} - (C_1 \cup C_2)$ can be partitioned into a collection B_2 of triples. Set

$$H = B_1 + B_2 + A_2.$$

It can be checked that in each of the above cases, $3(\bar{K}_v \vee K_{u-v}) \cup L(v,3) - H$ satisfies conditions (1)–(3) of Construction 2.4. This completes the proof. □

5. The case $\lambda = 5$ and $v \equiv 2 \pmod 6$

In this section, let $v \equiv 2 \pmod 6$, $v \geqslant 8$, $V(K_v) = V(\bar{K}_v) = X = \{a_i | 0 \leqslant i \leqslant v-1\}$, and let (X, A) be an MPT$(v,5)$ with leave $L(v,5) = \{\{a_0, a_i\} | 1 \leqslant i \leqslant 3\} \cup \{\{a_{2i}, a_{2i+1}\} | 2 \leqslant i \leqslant v/2 - 1\}$. Let $u > v$, $V(K_{u-v}) = Z_{u-v}$, $Y = X \cup Z_{u-v}$, $X \cap Z_{u-v} = \emptyset$ and $A_3 = \{\{0, a_0, a_1\}, \{1, a_0, a_2\}, \{2, a_0, a_3\}, \{a_1, 1, 2\}, \{a_2, 0, 2\}, \{a_3, 0, 1\}\}$.

Lemma 5.1. *Let $v \equiv 2 \pmod 6$, $v \geqslant 8$ and $u \equiv 3 \pmod 6$. If $u \geqslant 2v + 1$, then an MPT$(v,5)$ can be embedded in a TS$(u,5)$.*

Proof. Write $v = 6h + 8$ and $u - v = 6t + 7$. Suppose $1 \leqslant h + 1 \leqslant t \leqslant 3h + 2$. By Lemma 3.5, K_{u-v} contains 5 partial 1-factors F_0, F_1, F_2, F_3 and F each missing the three vertices of $\{0,1,2\}$ and $6h+4$ partial 1-factors $F_4, F_5, \ldots, F_{6h+6}$ and F_{6h+7} each missing exactly the vertex 2 and containing no edge $\{0,1\}$. Suppose $F = \{\{2i-1, 2i\} | 2 \leqslant i \leqslant 3t+3\}$, and let $B_1 = \{\{2, 2i-1, 2i\} | 2 \leqslant i \leqslant 3h+3\} \cup (\bigcup_{i=0}^{6h+7} a_i * F_i) \cup \{\{2, a_{2i}, a_{2i+1}\} | 2 \leqslant i \leqslant 3h+3\}$. Choose differences of $D(6t+7, 4)$ to form the following collection T of difference triples:

$$2\{(2t+1, 2t+4, 2t+2);\ (1, 2t, 2t-1);\ (2, 2t+1, 2t+3),$$
$$(2i, 3t-i+3, 3t-i+4) : 1 \leqslant i \leqslant t-1,\ t \geqslant 2,$$
$$(2i-1, t+i+1, t-i+2) : 1 \leqslant i \leqslant t-1,\ t \geqslant 2\}.$$

From $T \setminus \{(2t+1, 2t+4, 2t+2); (2t+1, 2t+4, 2t+2)\}$, choose $5t - 5h - 2$ difference triples $D_1, D_2, \ldots, D_{5t-5h-3}$ and $D_{5t-5h-2}$. Let $B_2 = (D_1) + (D_2) + \cdots + (D_{5t-5h-2})$, $B_3 = \{\{0, 2t+1, 4t+3\}, \{0, 2t+1, 4t+3\}\} + \{\{0, 2t+1, 4t+3\} + i | 1 \leqslant i \leqslant 2t - 2h\}$, the vertex set $V = \{i, 2t+i+1, 4t+i+3 | 1 \leqslant i \leqslant 2t-2h\}$. Clearly there exists a permutation σ of Z_{6t+7} such that $\sigma(0) = 0$, $\sigma(2t+1) = 1$, $\sigma(4t+3) = 2$ and $\sigma(V) = \{i | 6h+7 \leqslant i \leqslant 6t+6\}$. Set

$$H = B_1 + \sigma(B_2) + \sigma(B_3) + A_3.$$

Then the graph $5(\bar{K}_v \vee K_{u-v}) \cup L(v,5) - H$ satisfies conditions (1)–(3) of Construction 2.4. The conclusion follows. □

Lemma 5.2. *Let $v \equiv 2 \pmod 6$, $v \geqslant 8$ and $u \equiv 1 \pmod 6$. If $u \geqslant 2v + 1$, then an MPT$(v,5)$ can be embedded in a TS$(u,5)$.*

Proof. Write $v = 6h + 8$ and $u - v = 6t + 5$. Suppose $1 \leqslant h + 1 \leqslant t \leqslant 3h + 3$.

Case 1: $t = 1$. By Lemma 3.5, $2K_{11}$ contains 5 partial 1-factors F_0, F_1, F_2, F_3 and F each missing exactly the three vertices of $\{0,1,2\}$ and 4 partial 1-factors F_4, F_5, F_6 and F_7 each missing exactly the vertex 2 and containing no edge $\{0,1\}$. Choose two difference triples $(3,3,5)$ and $(2,2,4)$ from $D(11,3)$. Suppose $F = \{\{2i-1, 2i\} | 2 \leqslant i \leqslant 5\}$, and let $B_1 = \{\{2, 2i-1, 2i\} | 2 \leqslant i \leqslant 4\} \cup \{\{2, a_4, a_5\}, \{2, a_6, a_7\}\} \cup \bigcup_{i=0}^{7} a_i * F_i$, and $B_2 = \{\{0,3,6\} + i | 0 \leqslant i \leqslant 1, 5 \leqslant i \leqslant 10\} \cup \{\{0,2,4\} + i | 0 \leqslant i \leqslant 2, 5 \leqslant i \leqslant 10\}$. Set

$$H = A_3 + B_1 + B_2.$$

Case 2: $t > 1$. Let the partial 1-factors F_0, $F_1, \ldots, F_{6h+7}$ and F in K_{6t+5} be similar to that in the proof of Lemma 5.1, and $B_1 = (\bigcup_{i=0}^{6h+7} a_i * F_i) \cup \{\{2, 2i-1, 2i\} | 2 \leqslant i \leqslant 3h+4\} \cup \{\{2, a_{2i}, a_{2i+1}\} | 2 \leqslant i \leqslant 3h+3\}$. Choose differences of $D(6t+5, 4)$ to form the following collection T of difference triples:

$$2\{(2t+1, 2t+1, 2t+3);\ (2, 2t+2, 2t+4);\ (3, 2t-3, 2t),$$

$$(2t, 2t+2, 2t+3);\ (2i, 3t-i+2, 3t-i+3) : 1 \leqslant i \leqslant t-2,\ t \geqslant 3,$$

$$(2i-1, t+i+1, t-i+2) : 1 \leqslant i \leqslant t-2,\ t \geqslant 3\}.$$

From $T \setminus \{(2t+1, 2t+1, 2t+3); (2t+1, 2t+1, 2t+3)\}$, choose $5t-5h-5$ difference triples $D_1, D_2, \ldots, D_{5t-5h-6}$ and $D_{5t-5h-5}$. Let $B_2 = (D_1) + (D_2) + \cdots + (D_{5t-5h-5})$, $B_3 = \{\{0, 2t+1, 4t+4\} + i | 2h+2 \leqslant i \leqslant 6t+4\}$, $B_4 = \{\{0, 2t+1, 4t+3\} + i | 2t+1 \leqslant i \leqslant 6t+4\}$, the vertex set $V_1 = \{i, 2t+i+1, 4t+i+4 | 0 \leqslant i \leqslant 2h+1\}$, and the vertex set $V_2 = \{i, 2t+i+1, 4t+i+4 | 2h+2 \leqslant i \leqslant 2t\}$. Clearly $V_1 \cap V_2 = \phi$, so there exists a permutation σ of Z_{6t+5} such that $\sigma(V_1) = \{i | 3 \leqslant i \leqslant 6h+8\}$ and $\sigma(V_2) = \{i | 6h+9 \leqslant i \leqslant 6t+4\} \cup \{2\}$. Set

$$H = A_3 + B_1 + \sigma(B_2) + \sigma(B_3) + \sigma(B_4).$$

It can be checked that in each of the above cases $5(\bar{K}_v \vee K_{u-v}) \cup L(v, 5) - H$ satisfies conditions (1)–(3) of Construction 2.4. The conclusion then follows. □

6. Conclusion

Combining Lemmas 1.6, 4.1–4.3, 5.1–5.2, Theorems 1.1, 2.3, and using Construction 2.1, we have completely proved Theorem 1.7, the main theorem of this paper. Based on this result, recently we have completely solved the problem of embedding any MPT(v, λ) in an MPT(u, λ) in a subsequent paper, and we will continue to consider the embedding problem for simple maximum packings or the packings which are not maximum.

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References

[1] C.J. Colbourn, A. Rosa, Quadratic leaves of maximal partial triple system, Graphs Combin. 2 (1986) 317–337.
[2] C.J. Colbourn, A. Rosa, Triple Systems, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1999.
[3] J. Die Petersen, Theorie der regularen Graphen, Acta Math. 15 (1891) 193–220.
[4] Douglas B. West, Introduction to Graph Theory, Prentice-Hall, Inc., Upper Saddle River, NJ, 1996.
[5] J. Doyen, R.M. Wilson, Embeddings of Steiner triple systems, Discrete Math. 5 (1973) 229–239.
[6] H.L. Fu, C.C. Lindner, C.A. Rodger, Two Doyen-Wilson theorems for maximum packings with triples, Discrete Math. 178 (1998) 63–71.
[7] A. Hartman, Partial triple systems and edge colouring, Discrete Math. 62 (1986) 183–196.
[8] A. Hartman, E. Mendelsohn, A. Rosa, On the strong Lindner conjecture, Ars Combin. 18 (1983) 139–150.
[9] A.J.W. Hilton, C.A. Rodger, The embedding of partial triple systems when 4 divides λ, J. Combin. Theory (A) 56 (1991) 109–137.
[10] E. Mendelsohn, A. Rosa, Embedding maximal packings of triples, Congr. Numer. 40 (1983) 235–247.
[11] E. Mendelsohn, N. Shalaby, H. Shen, Nuclear designs, Ars Combin. 32 (1991) 225–238.
[12] S. Milici, G. Quattrocchi, H. Shen, Embedding of simple maximum packings of triples with λ even, Discrete Math. 145 (1995) 191–200.
[13] C.A. Rodger, S.J. Stubbs, Embedding partial system, J. Combin. Theory (A) 44 (1987) 241–253.
[14] H. Shen, Embeddings of simple triple systems, Sci. China Ser. A 35 (1990) 283–291.
[15] G. Stern, Triplesystem mit Untersystem, Arch. Math. 33 (1979) 204–208.